**Exercise 1**: The Distortion Energy Criterion of yielding assumes that yielding starts when the distortion energy at a point in a solid becomes equal to the distortion energy at yield in simple tension of the same material.

(1) Show that the energy of distortion per unit volume for the general loading can be expressed as,

$$W_d^*(\sigma_{ij}) = \frac{1+v}{6E}\left[\left(\sigma_{11}-\sigma_{22}\right)^2 + \left(\sigma_{22}-\sigma_{33}\right)^2 + \left(\sigma_{33}-\sigma_{11}\right)^2 + 6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)\right] \quad \text{(a)}$$

(2) If the yield stress of the material in uniaxial tension is $\sigma_Y$ show that this criterion is expressed as,

$$\sigma_Y = \frac{1}{\sqrt{2}}\left[\left(\sigma_{11}-\sigma_{22}\right)^2 + \left(\sigma_{22}-\sigma_{33}\right)^2 + \left(\sigma_{33}-\sigma_{11}\right)^2 + 6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)\right]^{1/2} \quad \text{(b)}$$

*Solution*

The strain energy density in terms of stresses is,

$$W^*(\sigma_{ij}) = \frac{1}{2E}\left[\sigma_{11}^2+\sigma_{22}^2+\sigma_{33}^2 - 2v\left(\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}\right)\right] + \frac{1}{2\mu}\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right) \quad \text{(c)}$$

$$W^*(\sigma_{ij}) = W_d^*(\sigma_{ij}) + W_p^*(\sigma_{ij}) \quad \text{(d)}$$

$$W_p^*(\sigma_{ij}) = 3\frac{1}{2}\left(\frac{1}{3}\varepsilon_{ii}\,\frac{1}{3}\sigma_{ii}\right) \quad \text{(e)}$$

From Hook’s law,

$$\varepsilon_{ii} = \frac{1-2v}{E}\sigma_{ii}$$

$$W_p^*(\sigma_{ij}) = 3\frac{1}{2}\left[\left(\frac{1}{3}\frac{1-2v}{E}\sigma_{ii}\right)\left(\frac{1}{3}\sigma_{kk}\right)\right] = \frac{1}{6}\frac{1-2v}{E}\sigma_{kk}\sigma_{ii} = \frac{1}{6}\frac{1-2v}{E}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)^2$$

$$W_d^*(\sigma_{ij}) = W^*(\sigma_{ij}) - W_p^*(\sigma_{ij})$$

$$W_d^*(\sigma_{ij}) = \frac{1}{2E}\left[\sigma_{11}^2+\sigma_{22}^2+\sigma_{33}^2 - 2v\left(\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}\right)\right] + \frac{1}{2\mu}\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right) - \frac{1-2v}{6E}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)^2 \quad \text{(f)}$$

$$W_d^*(\sigma_{ij}) = \frac{1}{2E}\left[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - 2v\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}\right)\right] - \frac{1-2v}{6E}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)^2$$
$$+\frac{1}{2\mu}\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)$$

$$W_d^*(\sigma_{ij}) = \frac{1}{6E}\left[3\sigma_{11}^2 + 3\sigma_{22}^2 + 3\sigma_{33}^2 - 6v\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}\right)\right] +$$
$$\frac{1}{6E}\left[-(1-2v)\left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}\right)\right)\right] + \frac{1}{2\mu}\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)$$

Combing the terms in the first two brackets, and replacing $\mu = E/2(1+v)$ we obtain

$$W_d^*(\sigma_{ij}) = \frac{1+v}{6E}\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2 + 6\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\right]$$

In uniaxial tension we have

$$W_d^*(\sigma_{ij}) = \frac{1+v}{6E}\left[2\sigma_{11}^2\right] = \frac{1+v}{6E}\left[2\sigma_Y^2\right]$$

Setting equal the right hand side of the two last expressions, we obtain (b).

**Exercise 2**: The stress state at a point of a solid is

$$\sigma_{ij} = \begin{pmatrix} \sigma & \tau & 0 \\ \tau & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

where $\sigma, \tau$ are given stress. What is the yield condition according to (a) Tresca and (b) V Mises criteria?

*Solution*

From the given stress matrix we can see that one of the principal stresses is $\sigma$ and the other two are

$$\sigma_{1,2} = \frac{\sigma + \sigma}{2} \pm \sqrt{\left(\frac{\sigma - \sigma}{2}\right)^2 + \tau^2} = \sigma \pm \tau$$

$$\Rightarrow \quad \sigma_1 = \sigma + \tau, \quad \sigma_2 = \sigma, \quad \sigma_3 = \sigma - \tau$$

(a) criterion

$$\sigma_1 - \sigma_3 = 2\tau = \sigma_Y \quad \Rightarrow \quad \tau = \sigma_Y / 2$$

(b) V Mises criterion

$$\sigma_Y = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \sqrt{3}\tau$$
$$\Rightarrow \tau = \sigma_Y / \sqrt{3}$$

**Exercise 3**: A thick-walled cylinder, with open ends, internal radius $r_i$ and external radius $2r_i$ is subjected to internal pressure $P_i$. The tensile yield stress of the material is $\sigma_Y$. Determine the internal pressure at the onset of yielding using the Tresca and V Mises yield criteria. Calculate the displacement at the onset of yielding at the internal surface of the cylinder (modulus of elasticity and Poisson ratio $E, v$ are known).

*Solution*

The maximum stresses are at the internal surface of the cylinder. They are given by,

$$\sigma_{rr} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e + \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$
$$\sigma_{\theta\theta} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e - \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

For $P_e = 0$ and the given radii, we obtain

$$\sigma_{rr} = \frac{1}{(2r_i)^2 - r_i^2}\left(P_i r_i^2 - \frac{P_i r_i^2 (2r_i)^2}{r_i^2}\right) = \frac{P_i r_i^2}{(2r_i)^2 - r_i^2}(1-4) = -P_i$$
$$\sigma_{\theta\theta} = \frac{1}{(2r_i)^2 - r_i^2}\left(P_i r_i^2 + \frac{P_i r_i^2 (2r_i)^2}{r_i^2}\right) = \frac{P_i r_i^2}{(2r_i)^2 - r_i^2}(1+4) = \frac{5}{3}P_i$$

They are also principal stresses.

V Misses criterion,

$$2\sigma_Y^2 = \left(-P_i - \frac{5P_i}{3}\right)^2 + \left(\frac{5P_i}{3}\right)^2 + (P_i)^2$$
$$2\sigma_Y^2 = \left(\frac{8P_i}{3}\right)^2 + \left(\frac{5P_i}{3}\right)^2 + (P_i)^2 \quad \Rightarrow P_i = \sqrt{\frac{18}{64+25+9}}\sigma_Y = 0.428\sigma_Y$$

Tresca Criterion

$$\sigma_{\theta\theta} - \sigma_{rr} = \frac{5}{3}P_i - (-P_i) = \sigma_Y \quad \Rightarrow P_i = \frac{3}{8}\sigma_Y = 0.375\sigma_Y$$

Displacement

$$u_r = \frac{r_i^2 P_i r}{E(r_e^2 - r_i^2)}\left[(1-v) + (1+v)\frac{r_e^2}{r^2}\right] = \frac{r_i^2 P_i r_i}{E((2r_i)^2 - r_i^2)}\left[(1-v) + (1+v)\frac{(2r_i)^2}{r_i^2}\right]$$
$$= \frac{P_i r_i}{3E}\left[(1-v) + 4(1+v)\right] = \frac{5+3v}{3}\frac{P_i r_i}{E}$$

**Exercise 4**: Express the plastic strain increment ratios for

(1) Simple tension $\sigma_{11} = \sigma_Y$

(2) Biaxial stress with $\sigma_{11} = -\sigma_Y / \sqrt{3},\ \sigma_{22} = \sigma_Y / \sqrt{3},\ \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{13} = 0$

(3) Pure shear $\sigma_{12} = \sigma_Y$

*Solution*

The plastic strain increment ratios are given by,

$$\frac{d\varepsilon_1^p}{s_1} = \frac{d\varepsilon_2^p}{s_2} = \frac{d\varepsilon_3^p}{s_3} = d\lambda \qquad \text{(Appendix C)} \qquad \text{(C.24a)}$$

Here $s_i \ \ (i = 1,2,3)$ are the principal values of the deviatoric stress tensor.

(1)

$$\sigma_{11} = \sigma_1 = \sigma_Y,\ \ \sigma_2 = \sigma_3 = 0$$

$$\Rightarrow s_1 = \sigma_Y - \frac{1}{3}\sigma_Y = \frac{2}{3}\sigma_Y, s_2 = s_3 = \sigma_2 - \frac{1}{3}\sigma_1 = -\frac{1}{3}\sigma_Y$$

$$\Rightarrow \frac{d\varepsilon_1^p}{2} = \frac{d\varepsilon_2^p}{-1} = \frac{d\varepsilon_3^p}{-1}$$

(2)

$$\sigma_{11} = \sigma_1 = \sigma_Y / \sqrt{3},\ \ \sigma_2 = 0,\ \ \sigma_3 = -\sigma_Y / \sqrt{3},$$

$$\Rightarrow s_1 = \sigma_Y / \sqrt{3},\ \ s_2 = 0,\ \ s_3 = -\sigma_Y / \sqrt{3}$$

$$\Rightarrow \frac{d\varepsilon_1^p}{1} = \frac{d\varepsilon_3^p}{-1}$$

The term with the second component is considered zero because the when the denominator is zero, the numerator is taken as zero in the theory.

(3) In simple shear we have $\sigma_{12} = \sigma_Y$

$$\sigma_1 = \sigma_Y,\ \ \sigma_2 = 0,\ \ \sigma_3 = -\sigma_Y \qquad \Rightarrow s_1 = \sigma_Y,\ \ s_2 = 0,\ \ s_3 = -\sigma_Y$$

$$\Rightarrow \frac{d\varepsilon_1^p}{1} = \frac{d\varepsilon_3^p}{-1}$$

**<u>Solutions of problems from a previous examination</u>**

**Problem A:**

**Stresses:** We have here a problem of plane stress. We assume the following stress filed,

$$\sigma_{11} = \sigma_{11}(x_2), \quad \sigma_{22} = \sigma_{12} = 0 \tag{a}$$

This stress field satisfies the equilibrium equations,

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0; \quad \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \tag{b}$$

The compatibility equation is,

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\sigma_{11} + \sigma_{22} + \alpha ET\right) = 0 \Rightarrow \frac{\partial^2}{\partial x_2^2}\left(\sigma_{11} + \alpha ET\right) = 0$$

Integrating we obtain, $\sigma_{11} + \alpha ET = c_1 x_2 + c_2 \Rightarrow \sigma_{11} = -\alpha ET + c_1 x_2 + c_2$ (c)

**Strains:** $\varepsilon_{11}(x_2) = \dfrac{\sigma_{11}(x_2)}{E} + \alpha T(x_2); \quad \varepsilon_{22} = \dfrac{-v\sigma_{11}(x_2)}{E} + \alpha T(x_2); \quad \varepsilon_{12} = 0$

**BC:** At the free ends we have the normal force and moment both equal zero,

$$\int_{-h}^{+h} \sigma_{11}(x_2) t dx_2 = 0; \quad \int_{-h}^{+h} \sigma_{11}(x_2) x_2 t dx_2 = 0 \tag{d}$$

Substituting (c) in (d) we obtain the constants,

$$c_1 = \frac{3}{2h^3}\int_{-h}^{+h} \alpha ET x_2 dx_2; \quad c_2 = \frac{1}{2h}\int_{-h}^{+h} \alpha ET dx_2 \quad \Rightarrow \sigma_{11} = \alpha E\left(-T + \frac{t}{2ht}\int_{-h}^{+h} T dx_2 + \frac{3tx_2}{2th^3}\int_{-h}^{+h} T x_2 dx_2\right)$$

Define $A = 2ht, \quad I_3 = \dfrac{2h^3 t}{3}$ then, $\Rightarrow \sigma_{11} = \alpha E\left(-T + \dfrac{t}{A}\displaystyle\int_{-h}^{+h} T dx_2 + \dfrac{tx_2}{I_3}\int_{-h}^{+h} T x_2 dx_2\right)$

Note that when $T = const$, the stress is zero because the second integral is zero and the first one is equal to –*T*).

**Problem B**:

1: Because of the parallel arrangement, $\sigma = \sigma_e + \sigma_v \,.\, \varepsilon = \varepsilon_e = \varepsilon_v$

For the spring $\sigma_e = E\varepsilon_e$; For the dashpot $\sigma_v = \eta \dfrac{d\varepsilon_v}{dt} = \eta \dot{\varepsilon}_v$

We add these stresses and set the strains equal we obtain for the Kelvin-Voigt model,

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

2: With the configuration of the elements in the Figure we can write,

$$\varepsilon_1 + \varepsilon_2 = \varepsilon \qquad \text{(a1)}$$

$$\sigma_1 = \sigma_2 = \sigma \qquad \text{(a2)}$$

$$\sigma_1 = \sigma = \eta_1 \dot{\varepsilon}_1 \quad \text{for the dashpot} \qquad \text{(b)}$$

$$\sigma_2 = \sigma = E_2 \varepsilon_2 + \eta_2 \dot{\varepsilon}_2 \quad \text{for the Kelvin-Voight} \qquad \text{(c)}$$

From (a1) and (b) we can write,

$$\dot{\varepsilon}_2 = \dot{\varepsilon} - \dot{\varepsilon}_1 = \dot{\varepsilon} - \frac{\sigma_1}{\eta_1}; \quad \Rightarrow \ddot{\varepsilon}_2 = \ddot{\varepsilon} - \frac{\dot{\sigma}_1}{\eta_1} \qquad \text{(d)}$$

Introduce (d) in (c) and rewrite,

$$\dot{\sigma} = E_2 \left( \dot{\varepsilon} - \frac{\sigma}{\eta_1} \right) + \eta_2 \left( \ddot{\varepsilon} - \frac{\dot{\sigma}}{\eta_1} \right) \quad \text{or} \quad \dot{\sigma} \left( \frac{\eta_1 + \eta_2}{E_2} \right) + \sigma = \frac{\eta_1 \eta_2}{E_2} \ddot{\varepsilon} + \eta_1 \dot{\varepsilon}$$